or, in expanded form,

$$\frac{df(X)}{da} = \frac{2X^5 + 6aX^4 + 3(1-b)X^3 + b(1-b)X + 2ab - 2bX}{X - a}. \tag{3}$$

The roots of $g(x) = 0$ satisfy the relations $0 < x_1 < a < x_2 < 2a$, from which we can show that the numerator on the right side of (3) is positive for both values of X. In fact, all the terms are positive except the last; but if $X = x_1 < a$, we have $2ab - 2bX > 0$, so that the numerator is positive in this case, while if $X = x_2$ we have, since $b < a^2$, and $a < x_2 < 2a$,

$$2X^5 + b(1-b)X - 2bX + 2ab = (2X^5 - b^2X) + (2ab - bX),$$
$$> (2a^4 - a^4)X + b(2a - 2a) > 0.$$

Since $x_1 - a$ is negative and $x_2 - a$ is positive, we thus have

$$\frac{df(x_1)}{da} < 0, \qquad \frac{df(x_2)}{da} > 0.$$

It follows that $f(x_1)$ has its greatest value, and $f(x_2)$ its least value, when $a = \sqrt{b}$, in which case $x_1 = x_2 = a$, and $f(x_1) = f(x_2) = 0$. The relations (1) follow at once.

II. We will show that the positive root of $f(x) = 0$ (there can be only one, by Descartes' Rule) lies between $x = \sqrt{b}$ and $x = a$. The interval $[\sqrt{b}, a]$ is, however, wholly comprised between the two roots of $g(x) = 0$, since $g(0)$ and $g(\infty)$ are positive, while

$$g(\sqrt{b}) = 2b - 2a\sqrt{b} = 2\sqrt{b}(\sqrt{b} - a)$$

is negative, and $g(a) = b - a^2$ is also negative.

We now prove that $f(\sqrt{b}) < 0$ and $f(a) > 0$. In the first place,

$$f(\sqrt{b}) = 2ab^2 + 2b\sqrt{b}(1-b) - 2ab.$$

Since $\sqrt{b} < a$, the substitution of a for $\sqrt{b}$ in the second term above gives

$$f\sqrt{b} < 2ab^2 + 2ab(1-b) - 2ab = 0.$$

On the other hand, we have

$$f(a) = 2a^5 + a^3(1-b) + ab(1-b) - 2ab = 2a^5 + a^3 - ab(a^2 + 1 + b).$$

If we replace each b above by a^2 we obtain the inequality

$$f(a) > 2a^5 + a^3 - a^3(a^2 + 1 + a^2) = 0.$$

Also solved by E. J. MOULTON, J. E. ROWE, H. H. CONWELL, J. A. BULLARD, J. L. RILEY, HORACE OLSON, J. W. BALDWIN, and the PROPOSER.

470. Proposed by ERNEST W. BROWN, Yale University.

There are n numbers each lying between $-\frac{1}{2}$ and $+\frac{1}{2}$, such that any value of each between these limits is equally probable. What is the probability that their sum will lie between $s - \frac{1}{2}$ and $s + \frac{1}{2}$, where s is an integral multiple of $\frac{1}{2}$.

SOLUTION BY C. F. Gummer, Kingston, Ontario.

Consider the more general problem where the n numbers are chosen at random from the intervals $(a_1, b_1), \cdots (a_n, b_n)$, and the sum is to lie in the interval (a, b).

Let $c_1 = b_1 - a_1, \cdots, c_n = b_n - a_n$.

Let $f_n(x)\cdot dx$ be the probability that the sum of the first r numbers will lie between x and $x + dx$.

Then $f_1(x) = 1/c_1$ when $a_1 < x < b_1$, and zero in other cases, and

$$f_r(x) = \int_{x-b_r}^{x-a_r} \frac{f_{r-1}(\xi)\cdot d\xi}{c_r}.$$

To evaluate $f_n(x)$, let us use the notations

$$\phi(x) = 1 \text{ for } x > 0, \qquad \phi(x) = -1 \text{ for } x < 0, \qquad \phi_r(x) = x^r\phi(x).$$

The function $\phi(x)$ is discontinuous but integrable. Since $\int_0^\alpha \phi(x)dx = \phi_1(\alpha)$, therefore, $\int_\alpha^\beta \phi(x)dx = \phi_1(\beta) - \phi_1(\alpha)$. In like manner,

$$\int_\alpha^\beta \phi_r(x)dx = \frac{\phi_{r+1}(\beta) - \phi_{r+1}(\alpha)}{r+1}.$$

Now

$$f_1(x) = \frac{1}{2c_1}\{\phi(x - a_1) - \phi(x - b_1)\}.$$

Hence,

$$f_2(x) = \frac{1}{c_2}\int_{x-b_2}^{x-a_2} f_1(\xi)d\xi$$

$$= \frac{1}{2c_1c_2}\{\phi_1(x - a_1 - a_2) - \phi_1(x - a_1 - b_2) - \phi_1(x - a_2 - b_1) + \phi_1(x - b_1 - b_2)\}.$$

We thus obtain by induction

$$f_n(x) = \frac{1}{2c_1 \cdots c_n \underline{|n-1}}\Sigma(-1)^k\phi_{n-1}(x - \alpha_1 - \beta_2 - \cdots - \kappa_n),$$

where, in the course of the summation, each of the letters $\alpha, \beta, \cdots, k$ is taken to represent either a or b independently of the others, and k is the number of them standing for b in any term.

The probability that the sum lies between a and b is then

(1) $$\int_a^b f_n(x)dx = \frac{1}{2c_1 \cdots c_n \underline{|n}}\Sigma(-1)^{k'-1}\phi_n(\omega - \alpha_1 - \cdots - \kappa_n),$$

where ω stands for a or b, and if it is b it is counted in k'.

We may modify the formula (1) as follows: If a is greater than $b_1 + b_2 + \cdots + b_n$ ($b > a$, $b_r > a_r$), the probability is zero. But the ϕ_n's now reduce to nth powers. Hence,

$$\frac{1}{2c_1c_2 \cdots c_n \underline{|n}}\Sigma(-1)^{k'-1}(\omega - \alpha_1 - \cdots - \kappa_n)^n = 0$$

for all values of a and b beyond certain limits, and is therefore identically zero. On adding this zero quantity to the right-hand side of (1) we cause all the ϕ_n's which are not mere nth powers to cancel, and double the others.

Hence, the probability is

$$\frac{1}{\underline{|n}\, c_1c_2 \cdots c_n}\Sigma(-1)^{k'-1}(\omega - \alpha_1 - \cdots - \kappa_n)^n,$$

the summation covering all the cases where $\omega - \alpha_1 - \cdots - k_n$ is positive.

It is easy to deduce for the special case propounded that the probability is

$$\frac{1}{\underline{|n}}\left\{\left(s + \frac{n+1}{2}\right)^n - {}_{n+1}C_1\left(s + \frac{n-1}{2}\right)^n + \cdots + (-1)^r{}_{n+1}C_r\left(s + \frac{n+1}{2} - r\right)^n\right\},$$

where r is the next integer below $s + (n + 1)/2$, and s may be any real number.

GEOMETRY.

Solutions of 497 were received from J. B. Reynolds, Elijah Swift, R. M. Mathews, and H. R Howard after selections for publication were made.

498. Proposed by FRANK R. MORRIS, Glendale, California.

To trisect an angle ABC, on BA and BC take D and E equidistant from B. Using DE as a diameter draw the semicircle $DFGE$. With the same radius and D and E as centers draw arcs locating the points F and G on this semicircle. Connect F and G with B. Prove that this method trisects a right angle and a straight angle and that it does not trisect an oblique angle.